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LETTER TO THE EDITOR

The fractal dimension and other percolation exponents in four and five dimensions

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Abstract. For percolation the fractal dimension d_f, which is identical to the magnetic field scaling power y_h, has never been calculated for hypercubic lattices of dimension $d=5$. Here we study percolation for systems of high dimensionality using the method of large-cell Monte Carlo position-space renormalisation group. We obtain the estimate $d_f=y_h=3.69\pm0.02$ ($d=5$) and 3.12 ± 0.02 ($d=4$). We also calculate the thermal scaling power $y_T=1/\nu$ where ν is the correlation length exponent. We find $y_T^{-1}=\nu=0.51\pm0.05$ ($d=5$) and $\nu=0.64\pm0.02$ ($d=4$). Finally, we compare our results with the ε expansions of d_f and ν.

How does one describe a percolation cluster in systems of high lattice dimensionality d? This question is particularly important since the critical dimensionality in percolation is given by $d_c=6$. Above $d=6$, the cluster structure is now reasonably well understood (see, e.g., Alexander *et al* 1984, Aharony *et al* 1984 and references therein). Roughly speaking, loops are irrelevant (i.e. the number of loops of size L drops off faster than a power law as L increases). The statistics of these clusters seems well described by the nodes and links picture, and the fractal dimension of the incipient infinite cluster is known to be given by the equation $d_f=(\gamma+\beta)/\nu$, where $\beta=\gamma=1$ are the order parameter and mean size exponents of a Cayley tree pseudo-lattice, and $\nu=\frac{1}{2}$ is the correlation length exponent. Since hyperscaling breaks down for $d>d_c$, conventional definition of the fractal dimension $d_f=d-\beta/\nu$ no longer holds (see Aharony *et al* 1984, Alexander *et al* 1984).

For $d=3$ and especially for $d=2$ there have been a wealth of studies of cluster structure (for a recent review, see Stanley and Coniglio 1983). It is by now generally accepted that the nodes and links picture of $d>6$ must be replaced by a somewhat more complex nodes/links/blobs picture where in addition to the link made of singly connected bonds there also appear relevant blobs made of multiply connected bonds. The blobs appear to be 'scaling quantities' whose mass decreases as a power law with system size (Hermann and Stanley 1984).

What happens for $d=4,5$? Essentially nothing is known—even the numerical values of d_f have not been calculated for $d=5$ and the estimate for $d=4$ (de Alcantara Bomfin *et al* 1980) is not very accurate. Our purpose here is to provide this much-needed information. We shall use the method of large-cell Monte Carlo PSRG (position-space

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renormalisation group), which has been demonstrated to provide reliable estimates for percolation exponents for $d=2$ (Reynolds *et al* 1980) and $d=3$ (Jan *et al* 1984).

In this method, a fraction p of the sites on a hypercubic lattice are occupied at random. The lattice is partitioned into cells of edge b containing b^d sites; the strategy is to consider a range of increasing values of b and then to extrapolate results to $b=\infty$.

In the original PSRG of Reynolds *et al*, a cell is considered to be occupied if a spanning cluster of neighbouring sites exists. Later, Lookman *et al* (1984) and Jan *et al* (1984) treated the cell as occupied when one passes through a maximum in the mean cluster size function χ. Jan and Stauffer (1984) compared the relative computational speed and found that the latter method was roughly five times faster than the former, if $d=2$; for $d>2$, this discrepancy is expected to further increase since the directions to be checked for a spanning cluster increases with d.

In this way we construct a histogram $L(p)$ giving the number of realisations that 'percolate' in the interval $[p, p+\mathrm{d}p]$. Hence the number of realisations that renormalise to occupied cells at concentration p is given by $R(p)=\int_0^p L(\bar{p})\,\mathrm{d}\bar{p}$. The renormalisation group is represented by $p'=R(p)$. The fixed points $p^*=R(p^*)$ include, in addition to the trivial values $p^*=0,1$, an additional non-trivial value which is identified with the percolation threshold p_c. The thermal scaling power y_T is obtained from the usual relation $y_T=\ln\lambda/\ln b$, where $\lambda=\mathrm{d}R/\mathrm{d}p|_{p=p^*}=L(p^*)$ is the 'thermal' eigenvalue.

Table 1 shows our calculations for a sequence of increasing cell sizes for $d=4,5$. The exponent ratio γ/ν is obtained from the finite-size scaling result that $\chi\sim b^{\gamma/\nu}$ (figure 1), from which $y_h=d_f$ is obtained using $d_f=\frac{1}{2}(\gamma/\nu+d)$ (table 2). To obtain the exponent $\nu=y_T^{-1}$, we must calculate the eigenvalue λ at the fixed point p^*. We find that our values p^* were indistinguishable from $\langle p_{\max}\rangle$, which is the value of p at which $\chi(p)$ has a maximum. To facilitate locating $\langle p_{\max}\rangle$, we approximate $L(p)$ by a Gaussian distribution

$$L(p)=(2\pi\sigma^2)^{-1/2}\exp[-\tfrac{1}{2}(p-\langle p_{\max}\rangle)^2/\sigma^2], \qquad (1)$$

Table 1. Results of the Monte Carlo renormalisation group for four- and five-dimensional hypercubic lattice. The symbols are defined in the text.

Four-dimensional random site percolation

Cell size b	$\chi_{\max}$	$\langle p_c(b)\rangle$	$(\langle p_c^2\rangle-\langle p_c\rangle^2)^{1/2}$	Trials	ν
10	8.70	0.2007	0.010 3021	1000	0.6297
15	22.11	0.1995	0.005 8384	1000	0.6411
20	42.72	0.1985	0.003 6580	1000	0.6385
25	67.74	0.1982	0.002 5239	1000	0.6358
30	101.90	0.1979	0.001 9107	1000	0.6368

Five-dimensional random site percolation

Cell size b	$\chi_{\max}$	$\langle p_c(b)\rangle$	$(\langle p_c^2\rangle-\langle p_c\rangle^2)^{1/2}$	Trials	ν
9	9.85	0.1445	0.005 8102	3000	0.5195
11	15.89	0.1443	0.003 5348	1000	0.5074
13	23.78	0.1430	0.002 6160	1000	0.5102
14	28.11	0.1426	0.002 3385	1000	0.5135
15	33.01	0.1425	0.001 9243	1000	0.5076

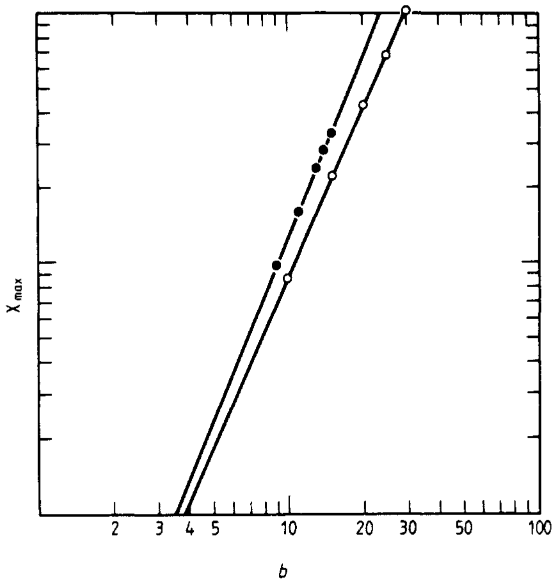

Figure 1. Dependence on cell-size b of $\chi_{\max}$, the maximum value of the mean cluster size. By finite size scaling this slope gives γ/ν.

Table 2. Results for γ/ν, y_h and y_T for four and five dimensions. These values are compared to the ε-expansion results. As figure 1 shows, corrections to scaling cannot be detected from our data.

Dimensionality	γ/ν	$y_h = d_f = \frac{1}{2}(\gamma/\nu + d)$	$y_T^{-1} = \nu$
6	2	4	0.5
5	2.38 ± 0.02	3.69 ± 0.02	0.51 ± 0.05
	$2.43^{+0.58}_{-0.48}$ [a]	3.72 ± 0.6 [a]	
	$2.48^{+0.41}_{-0.36}$ [a]	3.74 ± 0.4 [a]	
	2.07 [b]	3.54 [b]	0.56 [b]
4	2.24 ± 0.02	3.12 ± 0.02	0.64 ± 0.02
	$2.30^{+0.8}_{-0.62}$ [a]	3.21 ± 0.07 [a]	
	$2.43^{+0.32}_{-0.28}$ [a]	3.22 ± 0.3 [a]	
	2.18 [b]	3.09 [b]	0.62 [b]

[a] Stanley (1977).
[b] ε-expansion.

where $\sigma^2 = \langle p_{\max}^2 \rangle - \langle p_{\max} \rangle^2$. This function was found to faithfully represent $L(p)$ in the large b limit for $d = 4,5$ just as for $d = 2$ (Reynolds *et al* 1980, Stanley *et al* 1982).

Table 2 shows our final results for γ/ν, $y_h = d_f$ and $\nu = 1/y_T$ for $d = 6,5$ and 4. Figure 2 displays the variation of γ/ν and d_f with dimensionality and enables a visual comparison of our results with those of the ε expansion. We note that our data for d_f and ν are consistent with the ε expansion (Priest and Lubensky 1976, Amit 1976), but our values of γ/ν differ considerably. However the trend is the same since γ/ν increases as d decreases just below 6.

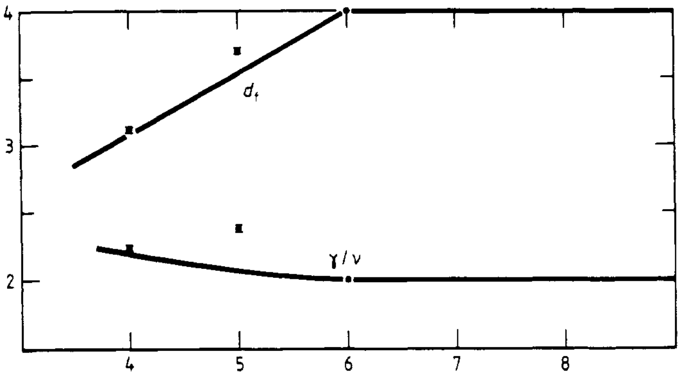

Figure 2. Dependence on d of d_f and γ/ν comparing our results with results of expansion in the parameter $\varepsilon = 6 - d$.

We conclude by noting that our estimates of y_h and y_T are sufficient to obtain expressions for the other static critical exponents. For example $\beta = (d - y_h)/y_T = 0.67$ ($d = 5$) and 0.56 ($d = 4$). Similarly $\gamma = (2y_h - d)/y_T = 1.21$ ($d = 5$) and 1.43 ($d = 4$), values which are consistent with the recent *series* estimates (Adler *et al* 1984) of $\gamma = 1.20$ ($d = 5$) and 1.44 ($d = 4$). Extrapolation of the data $p_c(b)$ to the $b \to \infty$ limit leads to $p_c = 0.141 \pm 0.001$ ($d = 5$) and 0.197 ± 0.001 ($d = 4$). These results are consistent with values reported by Schulte and Sprenger (1985) and have also been independently confirmed by Grassberger (1985).

Finally, to the extent that dynamic exponents in percolation are possibly related to static exponents, we may estimate dynamic exponents as well. For example, the Alexander-Orbach (AO) conjecture (Alexander and Orbach 1983) $d_w = \frac{3}{2} d_f$ permits us to obtain the conductivity exponent $t/\nu = \zeta/\nu + (d - 2)$ where $\zeta/\nu = d_w - d_f = \frac{1}{2} d_f$ and d_w is the fractal dimension of a diffusing particle on the percolating cluster. Thus we estimate $t/\nu = 4.85$ ($d = 5$) and 3.56 ($d = 4$). Our numerical values are fully consistent with AO and are also in good agreement with the recent results reported by Adler (1985) for t/ν: 4.77 ($d = 5$) and 3.52 ($d = 4$). An alternative conjecture, $d_w = 1 + d_f$ (Alexander 1983, Aharony and Stauffer 1984), is designed only for $d_f < 2$, fits badly for $d_f = 4$ (the Cayley tree) and seems also to fail for $d = 2$ (Havlin 1984, Stanley *et al* 1984). The conjecture of Sahimi (1984) $t/\nu = \nu^{-1} + 2(d - d_f)$ leads to 4.58 ($d = 5$) and 3.32 ($d = 4$). The AO predictions are in best agreement with the series results; however, there is recent evidence that the AO conjecture fails by about 3% for $d = 2$ (Zabolitzky 1984, Herrmann *et al* 1984, Hong *et al* 1984, Lobb and Frank 1984), but may be accurate in $d > 2$.

In conclusion, we have calculated the critical exponents and fractal dimensionality of random site percolation in four- and five-dimensional hypercubic lattices. We find agreement with some of the predictions of the ε expansion and good agreement with the recent results from the series expansion.

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